The Origins of Inequality: Insiders, Outsiders, Elites, and Commoners

Proofs of Formal Propositions

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Lemma 1. Note that $0 < s^a(x) < s^b(x)$ for all $x > 0$.

(i) Suppose $L(s, x) < d$. Part (a) in the definition of SRE implies $L(s, x) = (s/x)^{1/(1-\alpha)} < d$ and thus $s < s^a(x)$.

(ii) Suppose $L(s, x) = d$. The first order condition for part (b)(i) in the definition of SRE implies $\alpha s d^{\alpha-1} \leq x$ and thus $s \leq s^b(x)$. Part (b)(ii) in the definition of SRE implies $sd^\alpha - xd \geq 0$, and thus $s^a(x) \leq s$.

(iii) Suppose $L(s, x) > d$. The first order condition for part (b)(i) in the definition of SRE implies $L(s, x) = (\alpha s/x)^{1/(1-\alpha)} > d$ and thus $s^b(x) < s$.

(iv) Suppose $s < s^a(x)$. If $L(s, x) = d$ then result (ii) implies $s^a(x) \leq s$, which is a contradiction. If $L(s, x) > d$ then result (iii) implies $s^a(x) < s^b(x) < s$, which is a contradiction. Thus result (i) applies and (a) in Lemma 1 is true.

(v) Suppose $s^a(x) \leq s \leq s^b(x)$. If $L(s, x) < d$ then result (i) implies $s < s^a(x)$, which is a contradiction. If $L(s, x) > d$ then result (iii) implies $s > s^b(x)$, which is a contradiction. Thus result (ii) applies and (b) in Lemma 1 is true.

(vi) Suppose $s^b(x) < s$. If $L(s, x) < d$ then result (i) implies $s < s^a(x) < s^b(x)$, which is a contradiction. If $L(s, x) = d$ then result (ii) implies $s \leq s^b(x)$ which is a contradiction. Thus result (iii) applies and (c) in Lemma 1 is true.

Lemma 2. The "only if" statement follows from the definition of SRE. Here we assume $N = \int_0^1 L(s, x)ds$ and prove the "if" statement. Suppose in what follows that conditions (a), (b), and (c) from Lemma 1 are all satisfied.

(a) If $L(s, x) < d$, Lemma 1(a) implies $L(s, x) = (s/x)^{1/(1-\alpha)}$. This gives $x = sL(s, x)^{\alpha-1}$. Thus part (a) in the definition of SRE is satisfied for the given x.

(b) If $L(s, x) = d$, Lemma 1(b) implies $s^a(x) \leq s \leq s^b(x)$. The latter inequality gives $\alpha s d^{\alpha-1} \leq x$, which is the first order condition for $L = d$ to be a solution in part (b)(i) of the definition of SRE. This is sufficient for a maximum due to the concavity of the objective function. The former inequality gives $sd^\alpha - xd \geq 0$, so $r(s) \geq 0$ holds. Thus part (b)(ii) in the definition of SRE is also satisfied for the given x.

(c) If $L(s, x) > d$, Lemma 1(c) implies $L(s, x) = (\alpha s/x)^{1/(1-\alpha)}$ and $s^b(x) < s$. The latter inequality implies that $L(s, x)$ obeys the first order condition to be a solution in part (b)(i) of the definition of SRE. This is sufficient for a maximum due to the concavity of the objective function. Direct computation shows that $r(s) \geq 0$ also holds. Thus part (b)(ii) in the definition of SRE is also satisfied for the given x.

Proposition 1. For parts (a), (b), and (c), we first prove the implications from x to D(x). The implications from N to x will be established at the end of the proof.

(a) $x^a < x$ implies $1 < s^a(x)$. All sites are open. The result for D(x) is obtained by integrating the density from Lemma 1(a) on [0, 1].

(b) $x^b \leq x \leq x^a$ implies $s^a(x) \leq 1 \leq s^b(x)$. All sites are open or unstratified. The result for D(x) is obtained by integrating the density from Lemma 1(a) on $[0, s^a(x)]$ and the constant d from Lemma 1(b) on $[s^a(x), 1]$.

(c) $x < x^b$ implies $s^b(x) < 1$. Some sites are open, some are unstratified, and some are stratified. The result for D(x) is obtained by integrating the density from Lemma 1(a) on $[0, s^a(w)]$, the constant d from Lemma 1(b) on $[s^a(w), s^b(w)]$, and the density from Lemma 1(c) on $[s^b(w), 1]$.

(d) Continuity of the derivative at x^a and x^b can be verified by direct computations. Continuity of the derivative elsewhere is obvious. It can also be shown through computations that the derivative is always negative. The limiting values of D(x) follow from the results in parts (a) and (c).

(e) The existence of a unique $x > 0$ such that $D(x) = N$ follows from $N > 0$, the limits of D(x), the continuity of D(x), and the fact that D(x) is decreasing. The fact that this x and the associated density $L(\cdot, x)$ from Lemma 1 form a SRE follows from Lemma 2. The implicit function theorem shows that the equilibrium wage x(N) is continuously differentiable with $x'(N) < 0$. The limit results for x(N) follow from the limit results for D(x) in part (d).

Using the result for D(x) in part (b), we have $N^a = D(x^a)$ and $N^b = D(x^b)$, or equivalently $x^a = x(N^a)$ and $x^b = x(N^b)$. Because x(N) is decreasing, $N < N^a$ implies $x^a < x(N)$; $N^a \leq N \leq N^b$ implies $x^b \leq x(N) \leq x^a$; and $N^b < N$ implies $x(N) < x^b$. This completes the proof for parts (a), (b), and (c) above.

Proposition 2.

(a) From Proposition 1, $N < N^a$ implies $x^a < x$. This gives $1 < s^a(x)$ so all sites are open. The result for $\phi(x)$ is obtained using the density $L(s, x) = (s/x)^{1/(1-\alpha)}$ from Lemma 1(a) on [0, 1].

(b) From Proposition 1, $N^a \leq N \leq N^b$ implies $x^b \leq x \leq x^a$. This gives $s^a(x) \leq 1 \leq s^b(x)$ so that all sites are open or unstratified. The result for $\phi(x)$ is obtained using the density $L(s, x) = (s/x)^{1/(1-\alpha)}$ from Lemma 1(a) on $[0, s^a(x))$ and $L(s, x) = d$ from Lemma 1(b) on $[s^a(x), 1]$.

(c) From Proposition 1, $N^b < N$ implies $x < x^b$. This gives $s^b(x) < 1$ so that some sites are open, some sites are unstratified, and some sites are stratified. The result for $\phi(x)$ is obtained using the density $L(s, x) = (s/x)^{1/(1-\alpha)}$ from Lemma 1(a) on $[0, s^a(x))$; $L(s, x) = d$ from Lemma 1(b) on $[s^a(x), s^b(x)]$; and $L(s, x) = (\alpha s/x)^{1/(1-\alpha)}$ from Lemma 1(c) on $(s^b(x), 1]$.

Continuity of $\phi'(x)$ at x^a and x^b can be verified by computation. Continuity of $\phi'(x)$ for all other $x > 0$ is obvious. It can be shown by computation that $\phi'(x) < 0$ for all $x > 0$. Part (e) of Proposition 1 ensures that $x'(N)$ is continuous and negative for all $N > 0$. Together these results imply that $Y'(N)$ is continuous and positive for all $N > 0$.

<u>Corollary to Proposition 2</u>.

(a) From Proposition 1(a) and $N = D(x)$ we have $x(N) = (Q/N)^{1/(1-\alpha)}$. Using the solution for $\phi(x)$ from Proposition 2(a) along with (3) gives the result.

(b) From Proposition 1(b) and $N = D(x)$ we have $x(N) = (d-N)(2-\alpha)d^{\alpha-2}$. Using the solution for $\phi(x)$ from Proposition 2(b) along with (3) gives the result.

(c) Consider $N > N^b$ so that Proposition 2(c) applies. We have $Y'(N) = \phi'[x(N)]x'(N)$ and $Y''(N) = \phi''[x(N)][x'(N)]^2 + \phi'[x(N)]x''(N)$. From the identity $N \equiv D[x(N)]$,

$$x'(N) = 1/D'[x(N)] \qquad \text{and} \qquad x''(N) = -D''[x(N)]/\{D'[x(N)]\}^3.$$

Substituting these results into $Y''(N)$ gives

$$Y''(N) = -\{\phi'[x(N)]D''[x(N)] - \phi''[x(N)]D'[x(N)]\} / [D'(x(N))]^3.$$

Since $- 1/[D'(x(N))]^3 > 0$ from Proposition 1(d), the sign of $Y''(N)$ is the same as the sign of $\phi'[x(N)]D''[x(N)] - \phi''[x(N)]D'[x(N)]$. It therefore suffices to study the sign of $\phi'(x)D''(x) - \phi''(x)D'(x)$ on the interval $x < x^b$. Using Propositions 1(c) and 2(c), some algebra shows that this expression has the same sign as the quadratic $Av^2 + Bv + C$, where $v \equiv x^{1/Q}$; $A \equiv -d^{2(2-\alpha)}(1+\alpha)/\alpha^2$; $B \equiv d^{2-\alpha}\alpha^{\alpha/(1-\alpha)}(1-\alpha)^{-2}(2+\alpha-2\alpha^2)$; and $C \equiv -\alpha^{2/(1-\alpha)}(1-\alpha)^{-2}$. The quadratic is negative at $v = 0$ and positive at $v^b \equiv (x^b)^{1/Q}$. Since the quadratic is either rising throughout $[0, v^b)$ or has an interior maximum on this interval, there is a unique $v^c \in (0, v^b)$ at which the quadratic is zero, with a negative sign for all $v \in (0, v^c)$ and a positive sign for all $v \in (v^c, v^b)$. Thus there is a unique $x^c = (v^c)^Q \in (0, x^b)$ such that $\phi'(x)D''(x) - \phi''(x)D'(x) < 0$ for $x \in (0, x^c)$; $= 0$ for $x = x^c$; and > 0 for $x \in (x^c, x^b)$. Finally, this implies that there is a unique $N^c = N(x^c) > N^b$ such that $Y''(N) > 0$ for $N \in (N^b, N^c)$; $= 0$ for $N = N^c$; and < 0 for $N > N^c$.

(d) Continuous differentiability of $Y(N)/N$ follows from continuous differentiability of $Y(N)$, which was established in Proposition 2. We want to show that $Y(N)/N$ is globally decreasing. This can be done by direct computation for $0 < N \leq N^b$ using the results in parts (a) and (b) of the Corollary. Suppose $N > N^b$, which implies $x < x^b$ and $v < v^b$ in the notation used in the proof of (c) above. We will show that $Y'(N) - Y(N)/N < 0$ for all $N > N^b$. Using (3), the SRE identity $N \equiv D[x(N)]$, and the implicit function theorem, we obtain $Y'(N) - Y(N)/N = \phi'[x(N)]/D'[x(N)] - \phi[x(N)]/D[x(N)]$. Because $D'[x(N)] < 0$ from Proposition 1, this expression is opposite in sign to $\phi'[x(N)]D[x(N)] - \phi[x(N)]D'[x(N)]$. It thus suffices to show that the latter expression is positive for all $N > N^b$, or equivalently that $\phi'(x)D(x) - \phi(x)D'(x) > 0$ for all $x < x^b$. Some algebra shows that this is true iff $av^2 + bv + c > 0$ for all $v < v^b$ where $v \equiv x^{1/Q}$; $a \equiv d^{2(2-\alpha)}(1+\alpha)/2\alpha^{1/Q}$; $b \equiv d^{2-\alpha}(1/2 + \alpha - 1/\alpha)$; and $c \equiv \alpha^{1/(1-\alpha)}$. Since $a > 0$, this quadratic has a minimum value at $v_{min} = -b/2a$. There are three cases: (i) $v_{min} \leq 0$; (ii) $0 < v_{min} \leq v^b$; and (iii) $v^b < v_{min}$. In case (i), the quadratic is equal to $c > 0$ at $v = 0$ and positive for all $v > 0$. This yields the result. In case (ii), it suffices to show that the value of the quadratic is positive at v_{min}. This is true from $v_{min} \leq v^b$. In case (iii), the quadratic is positive at v^b, which implies that it is positive on $[0, v^b]$. This establishes that $Y(N)/N$ is decreasing as claimed. The result $\lim_{N\to 0} Y(N)/N = \infty$ follows from part (a) of the Corollary. To show that $\lim_{N\to\infty} Y(N)/N = 0$, note that $Y(N)/N = \phi[x(N)]/D[x(N)]$ where $N \to \infty$ implies $x \to 0$ from Proposition 1. Computing this ratio as a function of x using Proposition 1(c) for the denominator and Proposition 2(c) for the numerator gives the desired result.

<u>Proposition 3</u>.

(a) The SRE conditions from section 2 are built into the definition of $Y(N; \theta)$ from section 3. It suffices to show that for a fixed $\theta > 0$, there is a unique $N(\theta) > 0$ such that $Y[N(\theta); \theta]/N(\theta) = 1/\gamma$. This follows from the results in part (d) of the Corollary to Proposition 2.

(b) The LRE condition in (a) above and the implicit function theorem imply $N'(\theta) = Y_\theta[N(\theta), \theta] / \{Y[N(\theta); \theta]/N(\theta) - Y_N[N(\theta); \theta]\} > 0$ where the subscripts indicate partial derivatives. The inequality holds because the numerator is positive due to Proposition 2, and the denominator is positive due to part (d) of the Corollary. Continuity of $N'(\theta)$ follows from results in Proposition 2 and the Corollary. For the limits, note that (3) in the text and the definition of LRE together imply $1/\theta\gamma = \int_0^1 sL(s, x)^\alpha\, ds / D(x)$. Consider $\theta \to 0$, which implies that the left hand side $\to \infty$. Using Lemma 1 and Proposition 1, the right hand side $\to \infty$ iff $x \to \infty$. Therefore $\theta \to 0$ implies $x \to \infty$. By Proposition 1(e), this implies $N \to 0$. This establishes $\lim_{\theta\to 0} N(\theta) = 0$. The other limit result is obtained through similar reasoning.

(c) From Proposition 1, N^a and N^b are positive constants that do not depend on θ. Proposition 3(b) implies that there are unique productivity levels such that $N(\theta^a) = N^a$ and $N(\theta^b) = N^b$, with $0 < \theta^a < \theta^b$ because $0 < N^a < N^b$. The first sentences of (i), (ii), and (iii) are immediate from the fact that $N(\theta)$ is increasing. The second sentence of (i) results from the fact that $N < N^a$ implies $x^a < x$ due to Proposition 1, and thus $1 < s^a(x)$ in Lemma 1(a). The second and third sentences of (ii) result from the fact that $N^a \leq N \leq N^b$ implies $x^b \leq x \leq x^a$ due to Proposition 1, and thus $0 < s^a(x) \leq 1 \leq s^b(x)$ in Lemma 1(b). The second and third sentences of (iii) result from the fact that $N^b < N$ implies $x < x^b$ due to Proposition 1, and thus $0 < s^a(x) < s^b(x) < 1$ in Lemma 1(c). With minor notational abuse, define $s^a(\theta) \equiv s^a[x(N(\theta))]$ and $s^b(\theta) \equiv s^b[x(N(\theta))]$ as in Lemma 1. These functions are continuously differentiable because $N(\theta)$ and $x(N)$ are both continuously differentiable. They are decreasing because $N(\theta)$ is increasing and $x(N)$ is decreasing. The limit results follow from the limit results in part (b) above, the limit results in Proposition 1(e), and the definitions of $s^a(x)$ and $s^b(x)$.

(d) Continuous differentiability of $w(\theta) = \theta x[N(\theta)]$ follows from the continuous differentiability of $x(N)$ and $N(\theta)$. When $\theta \leq \theta^a$ we have $N \leq N^a$, $x^a \leq x$, and $1 \leq s^a$. All agents at all sites receive the food income w (including at s = 1 if $1 = s^a$ because the marginal site has zero rent). Thus LRE implies $w = Y(N; \theta)/N = 1/\gamma$. For the rest of the proof we assume $\theta^a < \theta$ so that $N^a < N(\theta)$. We need to show that $w'(\theta) = x[N(\theta)] + \theta x'[N(\theta)]N'(\theta) < 0$. Upon substituting the result from part (b) above for $N'(\theta)$, differentiating (3) to obtain the marginal product $Y_N(N, \theta)$, using the linearity of $Y(N; \theta)$ as a function of θ in Proposition 2 to eliminate the partial derivative $Y_\theta(N; \theta)$, and using the implicit function theorem to obtain $x'(N) = 1/D'[x(N)]$ from Proposition 1(e), a necessary and sufficient condition for the desired result is $\phi(x)[xD'(x) + D(x)] > x\phi'(x)D(x)$ for all relevant values of x. In the case where $\theta^a < \theta \leq \theta^b$ we have $N^a < N \leq N^b$ and $x^b \leq x < x^a$. Differentiating the functions from Propositions 1(b) and 2(b) yields an inequality involving a quadratic in x, which is satisfied for $x^b \leq x < x^a$. The other case is $\theta^b < \theta$, where we have $N^b < N$ and $x < x^b$. Differentiating the functions from Propositions 1(c) and 2(c) yields an inequality that does not involve x and holds whenever $\alpha < 1$.

<u>Proposition 4</u>. Using (4), we define z^a and z^b so that $s^a = s(z^a)$ and $s^b = s(z^b)$.

(a) From (7), $y_1(z) > y_2(z)$ for all $0 < z < 1$ implies $G_1 < G_2$, so it suffices to establish the first claim. We begin by considering the derivative $y'(z)$ of the Lorenz curve $y(z)$ in (6). The fraction of agents in the regional population who have the lowest income w is z^a

$\equiv (D_O + D_C)/N$ where D_O is the number of agents at open sites and D_C is the number of employed agents at stratified sites. The fraction of regional income going to this set of agents is $y^a \equiv w(D_O + D_C)/Y$. Thus for $z \in [0, z^a]$ we have $y'(z) = y^a/z^a = wN/Y$ and the Lorenz curve is linear.

For $z \in [z^a, 1]$ the derivative is $y'(z) = (N/Y)[w + r(s(z))/d]$ from (4), (5), and (6). The first derivative is continuous at z^a because $s(z^a) = s^a$ and $r(s^a) = 0$. For $z \in [z^a, z^b]$ the optimal labor input is $L(s) = d$ by Lemma 1(b), which gives $y'(z) = (\theta N/Y)d^{\alpha-1}s(z)$. From (4), this is linear and increasing in z. For $(z^a, z^b]$ we have $y''(z) = (\theta N^2/Y)d^{\alpha-2} > 0$ which is independent of z so that y(z) is quadratic on this interval. The second derivative $y''(z)$ is discontinuous at z^a, where it jumps from zero to a positive number.

Whenever the interval $(z^b, 1]$ is non-empty, the envelope theorem can be used to disregard effects operating through the optimal labor input L(s), and this yields $y''(z) = (\theta N^2/Yd^2)L[s(z)]^\alpha > 0$. This is larger than the second derivative on the quadratic interval $(z^a, z^b]$ because $L[s(z)] > d$ from Lemma 1(c), and it is increasing in z because s(z) and L(s) are both increasing. The first derivative $y'(z)$ is continuous at z^b because s(z) and r(s) are both continuous. Likewise $y''(z)$ is continuous at z^b because L(s) is continuous.

To compare the two Lorenz curves $y_1(z)$ and $y_2(z)$ from Proposition 4, we need to know how $y'(z)$ and $y''(z)$ respond to changes in N. We first show that the linear part of the Lorenz curve becomes flatter when N increases. For $N \in (N^a, N^b]$ the ratio wN/Y can be expressed in terms of x using Propositions 1(b) and 2(b). Differentiating with respect to x shows that wN/Y is increasing in x when a certain quadratic expression involving x is positive. This requirement satisfied whenever $x^b \leq x < x^a$, which follows from $N \in (N^a,$

N^b]. Since wN/Y is increasing in x, it is decreasing in N. For $N > N^b$, the ratio wN/Y can be expressed in terms of x using Propositions 1(c) and 2(c). Algebra and differentiation show that the ratio is increasing in x whenever $\alpha < 1$. Thus, the ratio is again decreasing in N. These results prove that $y_1'(z) > y_2'(z)$ whenever both curves are linear: that is, for the non-degenerate interval $0 \leq z \leq \min \{z_1^a, z_2^a\}$.

Now consider the slope of the Lorenz curve at z = 1: that is, $y'(1)$. When $N \in (N^a, N^b]$ we have $L(1) = d$ and $y'(1) = (\theta N/Y)d^{\alpha-1}$. The productivity parameter θ cancels with θ in the output expression $\phi(x)$ from Proposition 2(b), so this parameter does not affect the slope $y'(1)$. Part (d) of the corollary in section 3 shows that Y/N is decreasing in N so the ratio in parentheses is increasing in N. When $N > N^b$, using $L(1) = (\alpha/x)^{1/(1-\alpha)}$ from Lemma 1(c) gives $y'(1) = (\theta N/Yd)[xd + (1-\alpha)(\alpha/x)^{\alpha/(1-\alpha)}]$. As before, $\theta N/Y$ is increasing in N, and the value of θ does not affect $y'(1)$. The expression in brackets is decreasing in x whenever $x < x^b$, which follows from $N > N^b$. Therefore the expression in brackets is increasing in N, and $y'(1)$ is increasing in N. We observe that $y'(1)$ is continuous with respect to N at N^b. These results show that $y_1'(1) < y_2'(1)$, and by continuity that $y_1'(z) < y_2'(z)$ for all z in a non-degenerate neighborhood of z = 1.

By continuity, $y_1(z) - y_2(z)$ has a maximum value at some $z^* \in [0, 1]$. The results for $y_1'(z)$ and $y_2'(z)$ in the last two paragraphs together with $y_1(0) = y_2(0) = 0$ and $y_1(1) = y_2(1) = 1$ show that $y_1(z) - y_2(z)$ is strictly positive on a neighborhood of z = 0 and also on a neighborhood of z = 1. Thus for any maximizer z* we have $y_1(z^*) - y_2(z^*) > 0$ where z* is interior. Accordingly, there must be at least one local maximizer of $y_1(z) - y_2(z)$ on $z \in (0, 1)$ at which $y_1'(z^*) - y_2'(z^*) = 0$.

Suppose there is some interior point $z^0 \in (0, 1)$ at which $y_1(z^0) - y_2(z^0) = 0$ so the Lorenz curves intersect. This implies that there are at least two distinct interior local maxima separated by an interior local minimum. We will show that this is impossible.

First consider $N^a < N_1 < N_2 \leq N^b$. From previous results $y_1(z)$ is linear on $[0, z_1^a]$ and quadratic on $(z_1^a, 1]$. Likewise, $y_2(z)$ is linear on $[0, z_2^a]$ and quadratic on $(z_2^a, 1]$. It can be shown that z^a is a decreasing function of N on the interval $[N^a, N^b]$, which implies $0 < z_2^a < z_1^a$. Therefore $y_1'(z) - y_2'(z)$ is a positive constant on $[0, z_2^a]$, and it decreases on $(z_2^a, z_1^a]$ because y_2 becomes quadratic while y_1 remains linear. For $(z_1^a, 1]$, both y_1 and y_2 are quadratic. Previous results give $y'' = \theta N^2/Yd^{\alpha-2}$ whenever a Lorenz curve is quadratic. We have shown that the ratio $\theta N/Y$ does not depend on θ and is increasing in N. Thus y'' is increasing in N and $y_1''(z) < y_2''(z)$ on $(z_1^a, 1]$. This implies $y_1'(z) - y_2'(z)$ is decreasing on $(z_1^a, 1]$. Because $y_1'(z) - y_2'(z)$ is initially a positive constant and decreases thereafter, there cannot be more than one $z \in (0, 1)$ at which $y_1'(z) - y_2'(z) = 0$, so the Lorenz curves cannot intersect at an interior point. This proves part (a) for the case $N^a < N_1 < N_2 \leq N^b$.

Next suppose $N^a < N_1 \leq N^b < N_2$. As the preceding paragraph, $y_1(z)$ is linear on $[0, z_1^a]$ and quadratic on $(z_1^a, 1]$. Ignoring possible equalities among the boundaries z_1^a, z_2^a, and z_2^b, which do not affect the argument, there are three cases:

(i) $0 < z_1^a < z_2^a < z_2^b < 1$;

(ii) $0 < z_2^a < z_1^a < z_2^b < 1$; and

(iii) $0 < z_2^a < z_2^b < z_1^a < 1$.

For case (i), $y_1'(z) - y_2'(z)$ is initially a positive constant; then increases because y_1 becomes quadratic while y_2 remains linear; then decreases because both are quadratic (repeating a previous argument); and continues to decrease because y_1 remains quadratic while the second derivative of y_2 increases beyond quadratic. Thus there cannot be more than one $z \in (0, 1)$ at which $y_1'(z) - y_2'(z) = 0$.

For case (ii), $y_1'(z) - y_2'(z)$ is initially a positive constant; then decreases because y_1 remains linear while y_2 becomes quadratic; then decreases because both are quadratic; and continues to decrease because y_1 remains quadratic while the second derivative of y_2 increases beyond quadratic. Thus there cannot be more than one $z \in (0, 1)$ at which $y_1'(z) - y_2'(z) = 0$.

For case (iii), $y'_1(z) - y'_2(z)$ is initially a positive constant; then decreases because y_1 remains linear while y_s becomes quadratic; then decreases because y_1 remains linear while the second derivative of y_s increases beyond quadratic; and then decreases because y_1 is quadratic while y_2 is beyond quadratic (note in the last case that $y_1'(z) - y_2'(z)$ would be decreasing if both functions were quadratic, so this must also be true when the second derivative of y_2 is even larger). Thus there cannot be more than one $z \in (0, 1)$ at which $y'_1(z) - y'_2(z) = 0$. We have therefore shown that the Lorenz curves cannot intersect at an interior point in any of cases (i), (ii), or (iii) when $N^a < N_1 \leq N^b < N_2$. This completes the proof of part (a) in Proposition 4.

(b) To compute the Gini coefficient in (7), we need to compute $\int_0^1 y(z)\,dz$. From (6), this has two components, one involving the wage and the other involving land rent. The integral involving the wage is wN/2Y. Using Propositions 1(b) and 2(b) along with $N^b = 2Qd$ and $x^b = \alpha d^{\alpha-1}$ gives $wN/2Y = 2\alpha / (2+\alpha+\alpha^2)$ when this integral is evaluated at N^b. Using Propositions 1(c) and 2(c), wN/2Y approaches $\alpha/2$ as $N \rightarrow \infty$ and $x \rightarrow 0$.

To compute the rent component of $\int_0^1 y(z)\,dz$ in the insider-outsider range where $N^a < N \le N^b$ and $x^b \le x < x^a$, we first calculate $R(z) = \int_0^{s(z)} r(s)\,ds$ for $z^a \le z \le 1$ as in (5). Because no sites are stratified, $r(s) = 0$ for $0 \le s \le s^a$ and $r(s) = \theta(sd^\alpha - xd)$ for $s^a \le s \le 1$. We then compute $\int_0^1 R(z)\,dz$ where $R(z) = 0$ for $0 \le z \le z^a$ and $R(z) = \int_0^{s(z)} r(s)\,ds$ for $z^a \le z \le 1$. Dividing by $\phi(x)$ from Proposition 2(b) gives $\int_0^1 R(z)\,dz / Y$. Using $N^b = 2Qd$ and $x^b = \alpha d^{\alpha-1}$ this ratio is $\int_0^1 R(z)\,dz / Y = (2-\alpha)^2[(1-\alpha^3)/3(1-\alpha) - \alpha] / 2(1-\alpha)(2+\alpha+\alpha^2)$ at N^b. Adding this to the result in the preceding paragraph for wN/2Y to get $\int_0^1 y(z)\,dz$ at N^b and then using (7) to compute the Gini coefficient yields the result for $G(N^b)$ stated in Proposition 4(b). This is the upper bound for insider-outsider inequality because part (a) showed that the Gini is increasing in N on the interval $(N^a, N^b]$. It is the lower bound for elite-commoner inequality because part (a) showed that there is a relationship of Lorenz curve dominance between N^b and any $N_2 > N^b$.

To compute the rent component of $\int_0^1 y(z)\,dz$ in the elite-commoner range where $N^b < N$ and $x < x^b$, we first calculate $R(z) = \int_0^{s(z)} r(s)\,ds$ for $z^a \le z \le 1$ as in (5). Because some sites are now stratified, $r(s) = 0$ for $0 \le s \le s^a$; $r(s) = \theta(sd^\alpha - xd)$ for $s^a \le s \le s^b$; and $r(s) = \theta[s(\alpha s/x)^{\alpha/(1-\alpha)} - x(\alpha s/x)^{1/(1-\alpha)}]$ for $s^b < s \le 1$. We then compute $\int_0^1 R(z)\,dz$ where $R(z) = 0$ for $0 \le z \le z^a$ and $R(z) = \int_0^{s(z)} r(s)\,ds$ for $z^a \le z \le 1$. Dividing by $\phi(x)$ from Proposition 2(c) gives $\int_0^1 R[s(z)]\,dz / Y$. Letting $N \to \infty$ and $x \to 0$, it can be shown that this ratio approaches zero. Combining this with the earlier result $wN/2Y \to \alpha/2$ implies $\int_0^1 y(z)\,dz \to \alpha/2$. Using (7) to compute the Gini coefficient yields the limit result stated in Proposition 4(b).

Proposition 5.

Landless agents fail to replace themselves because $w < 1/\gamma$ from Proposition 3(d). The inequality $\theta^a < \theta$ gives $N > N^a$ from Proposition 3 and $x < x^a$ from Proposition 1. The latter result gives $s^a < 1$. The inequality $s^a < s^r$ follows from $w < 1/\gamma$ and $x = w/\theta$.

Suppose $\theta^a < \theta \leq \theta^b$ so that $N^a < N \leq N^b$, $x^b \leq x < x^a$, and $s^a < 1 \leq s^b$ (no sites are stratified). We want to show that $s^r < 1$. This is true if $1/\gamma = Y(N)/N < \theta d^{\alpha-1}$ where the equality follows from the definition of LRE. We can express the required inequality in terms of x as $\phi(x)/D(x) < \theta d^{\alpha-1}$ where the ratio on the left is obtained from Propositions 1(b) and 2(b). This reduces to a quadratic expression in x that must be positive. It can be shown that the latter expression is decreasing on $[x^b, x^a)$ and zero at x^a. This gives $s^r < 1$.

Suppose instead that $\theta^b < \theta$ so that $N^b < N$, $x < x^b$, and $s^a < s^b < 1$ (some sites are stratified). We want to show that $s^r < s^b$. This is true if $1/\gamma = Y(N)/N < \theta x/\alpha$ where the equality follows from the definition of LRE. We can express the required inequality in terms of x as $\phi(x)/D(x) < \theta x/\alpha$ where the ratio on the left is obtained from Propositions 1(c) and 2(c). Some algebra shows that this is true when $\alpha < 1$. This gives $s^r < s^b$.

The result $s^r \in (s^a, s^b)$ implies $L(s^r) = d$ from Lemma 1. By the definition of SRE in section 2 and the definition of s^r in Proposition 5, each insider at s^r has the income $w + r(s^r)/d = 1/\gamma$. This implies that these insiders exactly replace themselves. The remainder of Proposition 5 follows from the fact that r(s) is continuous and increasing.